Sequences - Algorithms

General overview

EXERCISE 1

Let (u_n) be a sequence such that : $u_0 = 1$ and for all n, $u_{n+1} = 3u_n - 1$.

a) Calculate u_1, u_2 and u_3 by hand. Express u_{n+2} as a function of u_n.

b) Write an algorithm in pseudocode given the term u_n, n given. Then give the values of u_5, u_{10} et u_{15}.

c) Write an algorithm given the first 10 terms of the sequence (u_n).

EXERCISE 2

Let (u_n) be a sequence defined by : $\begin{cases} u_0 = 2, \quad u_1 = 4 \\ u_{n+2} = 4u_{n+1} - u_n \end{cases}$

a) Calculate the terms u_2, u_3 and u_4 by hand.

b) Write an algorithm to calculate the nth term of the sequence. Calculate u_6 and u_{10} using this algorithm.

Monotonicity of a sequence

EXERCISE 3

Determine the monotonicity of the following sequences defined on $\mathbb{N}$:

a) $u_n = -3n + 1$ b) $u_n = \dfrac{n+1}{n+2}$ c) $u_n = 2^n$ d) $u_n = \left(-\dfrac{1}{2}\right)^n$

EXERCISE 4

Show that the sequence (u_n) is decreasing for $n \geqslant 2$: $u_n = \dfrac{n^2}{n!}$

$n! =$ factorial n and $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

EXERCISE 5

Determine the monotonicity of the following sequences :

a) $u_n = \dfrac{n^2}{2^n}, \quad n \geqslant 4$ b) $u_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^n}, \quad n \in \mathbb{N}$

EXERCISE 6

Show that the following sequence is decreasing : $u_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^n} - n$

Exercise 7

For each affirmation, say whether it is true or false. Justify your answer.

a) **Proposition 1 :** (u_n) and (v_n) are two increasing sequences, the sequence $w_n = u_n + v_n$ is also increasing.

b) **Proposition 2 :** (u_n) and (v_n) are two increasing sequences, the sequence $t_n = u_n \times v_n$ is also increasing.

Arithmetic and geometric sequences

Exercise 8

Let (u_n) be an arithmetic sequence with a common difference of r.

a) Express u_n in terms of n if $u_0 = 2$ and $r = \dfrac{1}{2}$

b) $u_2 = 41$ and $u_5 = -13$. Calculate u_{20}

c) $u_1 = -2$ and $r = 3$. Calculate u_{20} then $S = u_1 + u_2 + \cdots + u_{20}$

d) $u_0 = -3$ and $r = -2$. Calculate u_{25} and u_{125} then $S = u_{25} + u_{26} + \cdots + u_{125}$

Exercise 9

Let (u_n) be a sequence defined by $u_0 = 1$ and for all natural numbers n by : $u_{n+1} = \dfrac{u_n}{1 + u_n}$

a) Calculate u_1, u_2, u_3, u_4. What conjecture can be made with regards to the expression of u_n in terms of n ?

b) Show that the sequence (v_n) defined by $v_n = \dfrac{1}{u_n}$ is arithmetic.

c) Express v_n then u_n in terms of n.

Exercise 10

(u_n) is a geometric sequence with a common ratio of q.

a) $u_1 = 5$ and $q = \dfrac{2}{3}$. Express u_n in terms of n

b) $u_4 = 1$ and $u_9 = 25\sqrt{5}$. Calculate q then u_{14}

c) $q = 2$ and $S = u_0 + u_1 + \cdots + u_{12} = 24\,573$. Calculate u_0.

Exercise 11

Prove the sequence (u_n) defined by $u_n = \dfrac{2^n}{3^{n+1}}$ is geometric. Does it converge ?

Exercise 12

Calculate the following sums then check your result using an algorithm :

a) $\text{A} = 8 + 13 + 18 + \cdots + 2008 + 2013$

b) $\text{B} = \dfrac{1}{2} + 1 + \dfrac{3}{2} + 2 + \dfrac{5}{2} + \cdots + 10$

c) $\text{C} = 0,02 - 0,1 + 0,5 - 2,5 + \cdots + 312,5$

Arithmetico-geometric and homographic sequences

EXERCISE 13

Consider the sequence (u_n) defined by :

$$u_0 = 1 \quad \text{and for all natural numbers } n \quad u_{n+1} = \frac{1}{3}u_n + 4$$

Let v_n be a sequence defined by, $v_n = u_n - 6$

a) For all natural numbers n , express v_{n+1} in terms of v_n.
   What is the nature of the sequence (v_n) ?

b) Express v_n then u_n in terms of n.

c) Study the convergence of the sequence (u_n).

EXERCISE 14

An animal reserve has an initial population of 1 000 animals. This population changes each year because :

- 20 % of the animals disappear each year (overall balance of births and deaths)
- 120 animals a year are introduced into the reserve.

The purpose of this exercise is to determine how this population changes after n years (we will denote the population p_n with $p_0 = 1\,000$).

1) a) Determine a relationship between p_{n+1} and p_n.
   b) Conjecture graphically using a calculator how the population changes.

2) To prove this conjecture, we introduce an auxiliary sequence (v_n) such that : $v_n = p_n - 600$
   a) Show that the sequence (v_n) is geometric.
   b) Express v_n then p_n in terms of n.
   c) Does the sequence p_n admit a limit at $+\infty$? What conclusion can be made ?

EXERCISE 15

Consider (u_n) defined by : $u_0 = 0$ and $u_{n+1} = \dfrac{2u_n + 3}{u_n + 4}$

a) Let $v_n = \dfrac{u_n - 1}{u_n + 3}$. Show that the sequence (v_n) is geometric.

b) Express v_n then u_n in terms of n.

c) Determine the limit of (v_n) then that of (u_n).

EXERCISE 16

Antilles-Guyane sept 2010

Consider the sequence of real numbers (u_n) defined on $\mathbb{N}$ by :

$$u_0 = -1,\ u_1 = \frac{1}{2} \ \text{ and for all natural numbers } n,\ \ u_{n+2} = u_{n+1} - \frac{1}{4}u_n.$$

1) Calculate u_2 and deduce that the sequence (u_n) is neither arithmetic nor geometric.

2) Let (v_n) be a sequence defined by : $v_n = u_{n+1} - \frac{1}{2}u_n$.

   a) Calculate v_0.
   b) Express v_{n+1} in terms of v_n.
   c) Show that the sequence (v_n) is geometric with a common ratio of $\frac{1}{2}$.
   d) Express v_n in terms of n.

3) Let (w_n) be the sequence defined by : $w_n = \frac{u_n}{v_n}$

   a) Calculate w_0.
   b) Using the equality $u_{n+1} = v_n + \frac{1}{2}u_n$, express w_{n+1} in terms of u_n and of v_n.
   c) Show for all natural numbers n, $w_{n+1} = w_n + 2$.
   d) Express w_n in terms of n.

4) Show for all natural numbers n : $u_n = \frac{2n-1}{2^n}$

5) For all natural numbers n, let : $S_n = \sum_{k=0}^{k=n} u_k = u_0 + u_1 + \cdots + u_n$.

   Write an algorithm to calculate S_n for all n in $\mathbb{N}$. Then give the approximate values to 10^{-4} of S_6, S_{10} and S_{15}.
   What conjecture regarding the convergence of the sequence (S_n) can be made ?

   Note : We will prove this conjecture in the next chapter.

Exercise 17

2009 National sample

Consider the sequence (w_n) for all natural numbers $n \geqslant 1$:

$$nw_n = (n+1)w_{n-1} + 1 \quad \text{et} \quad w_0 = 1$$

The following table shows the first ten terms of the sequence.

w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
1	3	5	7	9	11	13	15	17	19

1) Itemize the calculation to obtain w_{10}.
2) What can we conjecture about the nature of the sequence (w_n) ? Calculate w_{2009} using this conjecture.

Exercise 18

Sum of squares

We intend to show that : $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

1) Determine a cubic polynomial P such that for all real numbers x we have : $P(x+1) - P(x) = x^2$

   Note : Write $P(x) = ax^3 + bx^2 + cx + d$ and determine the value of the coefficients a, b, c and d through a system of equations.

2) Complete these equalities

$$\begin{aligned} P(1) - P(0) &= \\ P(2) - P(1) &= \\ P(3) - P(2) &= \\ \dots \quad & \dots \\ P(n+1) - P(n) &= \end{aligned}$$

3) Deduce the formula for the sum of squares

Algorithms

Exercise 19

The function f defined on $\mathbb{R}$ by : $f(x) \begin{cases} -x+2 & \text{if } x < 1 \\ x^2 - 2x + 2 & \text{else} \end{cases}$

Write an algorithm which prints the value of $f(x)$ for a given x.

Exercise 20

The following algorithm is used to determine the linear coefficient of a line passing through two points.

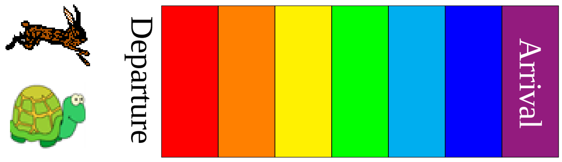
Variables: a, b, c, d, m real numbers
Inputs and initialization
  Print "enter the coordinates of a point A"
  Lire a, b
  Print "enter the coordinates of point B"
  Read c, d
Processing
  $\dfrac{d-b}{c-a} \rightarrow m$
Sorties: Print m

a) Modify this algorithm in order to print the constant coefficient of this line.
b) Enter this algorithm into your calculator
c) This algorithm does not take into account the case of a line parallel to the y-axis. Modify the algorithm for this case to be processed.

Exercise 21

The tortoise and the hare

This is a game that is played with dice on a board of seven boxes :



The rules of the game follow the algorithm opposite.

Note : T and L represent the respective positions of the tortoise and the hare.

1) Write the rules of the game as a short text.
2) Enter the algorithm into your calculator, denoting the hare and the tortoise with numbers.
3) Alter and complete this algorithm in order to simulate the game 1 000 times.
4) Does one of the two protagonists have an advantage using these rules ? If so, modify the number of squares on the board to make the game as fair as possible.

> **Variables**: T, L, D integers
> G text
> **Inputs and initialization**
> $0 \to T$
> $0 \to L$
> **Processing**
> **while** $T < 7$ and $L \neq 7$ **do**
> D takes the value of a die roll
> **if** $D = 6$ **then**
> $L = 7$
> $G =$ "Hase"
> **else**
> $T = T + D$
> **end**
> **if** $T \geqslant 7$ **then**
> $G =$ "‘Tortoise"
> **end**
> **end**
> **Output** : Print « The winner is : » G

Exercise 22

Consider the following algorithm.

1) Justify for $n = 3$, that the display is 11 for u and 21 for S
2) Copy and complete the following table :

n	0	1	2	3	4	5
u				11		
S				21		

> **Variables**: n, i integers
> u, S real numbers
> **Inputs and initialization**
> Read n
> $1 \to u$, $1 \to S$ et $0 \to i$
> **Processing**
> **while** $i < n$ **do**
> $2u + 1 - i \to u$
> $S + u \to S$
> $i + 1 \to i$
> **end**
> **Sorties**: Print u, S

Let (u_n) be the sequence defined by $u_0 = 1$ and $u_{n+1} = 2u_n + 1 - n$
Let (S_n) be the sequence defined by $S_n = u_0 + u_1 + \cdots + u_n$

3) Copy and complete the following table :

n	0	1	2	3	4	5
u_n	1					
$u_n - n$	1					

What conjecture can be made from the results of this table ?

4) Prove that : $u_n = 2^n + n$. Deduce the expression of S_n in terms of n.