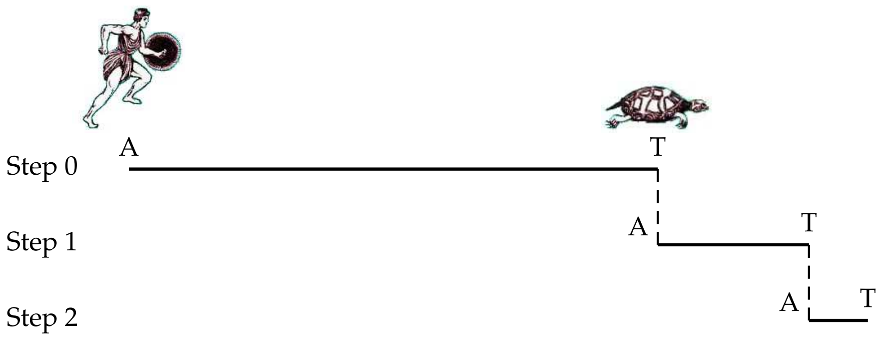
The paradox of Achilles and the tortoise

1 The paradox

The paradox of Achilles and the tortoise, devised by Zeno of Elea, says that one day, the Greek hero Achilles played a footrace against the slow reptile. As Achilles was deemed to be a very fast runner, he graciously granted the tortoise a head start of one hundred meters.
Zeno stated that if the tortoise has a head start then Achilles cannot catch up, no matter how fast he runs. By the time Achilles gets to the point where the tortoise had been, the tortoise has moved forward. So Achilles can never outrun the tortoise !

2 Resolution

Achilles can only catch up with the tortoise after an infinite number of paces. The paradox arises from saying that Achilles makes an infinite number of paces in an infinite amount of time.
Consider the following example to solve the paradox : Achilles runs at a speed of 10 ms^{-1}, which makes him a very good 100 meter sprinter, and the tortoise crawls at 0.1 ms^{-1} which is a 100 times slower than Achilles.
Let's have a closer look at the race :



With each pace, Achilles travels 100 times further than the tortoise because he is moving 100 times faster, therefore covering the distance AT 100 times faster previously. The time t_n taken to run n paces is :

$$t_n = 10 + \frac{10}{100} + \frac{10}{100^2} + \cdots + \frac{10}{100^{n-1}}$$

t_n is the sum of n first terms of a geometric progression with a common ratio of $\dfrac{1}{100}$ and a first term of 10. Therefore :

$$t_n = 10 \times \frac{1 - \dfrac{1}{100^n}}{1 - \dfrac{1}{100}} = \frac{1000}{99}\left(1 - \frac{1}{100^n}\right)$$

as $\displaystyle\lim_{n \to +\infty} \frac{1}{100^n} = 0$ because $-1 < \dfrac{1}{100} < 1$

Using the sum and product of limits : $\displaystyle\lim_{n \to +\infty} t_n = \frac{1000}{99} \simeq 10,1010$

Achilles takes a little more than 10,10 s to make an infinite number of paces. So Achilles catches up the tortoise, as well as anybody might expect !

3 Conclusion

It is easy to explain the paradox that an infinite number of paces can be made in finite time with the concept of a limit.
We can transpose this paradox to many common phenomena. For example, a ball that bounces to 80 % of its original height. It will bounce an infinite number of times in a finite time.