Mathematical induction. Limits of sequences

Mathematical induction

EXERCISE 1

Prove that for any natural number n, $4^n + 5$ is a multiple of 3.

EXERCISE 2

Prove that for any natural number n, $3^{2n} - 1$ is a multiple of 8.

EXERCISE 3

Is it true that for all natural numbers $n \geqslant 1$, $n^3 + 2n$ is a multiple of 3 ?

EXERCISE 4

Show that $\forall n \in \mathbb{N}$, $3^{2n+1} + 2^{n+2}$ is a multiple of 7.

EXERCISE 5

Consider $S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$ where $n \geqslant 1$. *Sum of squares*

a) Calculate S_1, S_2, S_3 and S_4. Express S_{n+1} in terms of S_n.

b) Prove by induction that for all natural numbers $n \geqslant 1$: $S_n = \dfrac{n(n+1)(2n+1)}{6}$

EXERCISE 6

Consider : $S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$ where $n \geqslant 1$ *Sum of cubes*

a) Calculate S_1, S_2, S_3 and S_4. Express S_{n+1} in terms of S_n.

b) Prove by induction that for all natural numbers $n \geqslant 1$: $S_n = \dfrac{n^2(n+1)^2}{4}$

EXERCISE 7

We note $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ where $n \geqslant 1$
Prove by induction that for all natural numbers non zero : $n! \geqslant 2^{n-1}$

EXERCISE 8

Let (u_n) be a sequence defined by : $u_0 \in]0;1[$ et $u_{n+1} = u_n(2 - u_n)$.

Prove by induction that : $\forall n \in \mathbb{N}, \quad 0 < u_n < 1$

Hint : Determine whether the function f defined by : $f(x) = x(2-x)$ is monotonic

EXERCISE 9

Let (u_n) be a sequence defined by : $u_0 = 1$ et $u_{n+1} = \sqrt{2 + u_n}$.

Prove by induction that for all natural numbers n, $0 < u_n < 2$ and (u_n) is increasing.

Exercise 10

Let (u_n) be a sequence defined for all $n \in \mathbb{N}$ by : $\begin{cases} u_0 = 1 \ , \ u_1 = 2 \\ u_{n+2} = 5u_{n+1} - 6u_n \end{cases}$

Prove by induction that for all $n \in \mathbb{N}$: $u_n = 2^n$

⚠ It takes two terms to initialize this property.

Exercise 11

Object of the question : we don't know the expression of u_n in terms of n. By calculating the first few terms a conjecture can be made as to the expression of u_n in terms of n. Then we prove this conjecture.

The sequence (u_n) is defined by : $u_1 = 0$ and $u_{n+1} = \dfrac{1}{2 - u_n}$

a) Calculate u_2 ; u_3 ; u_4 ; u_5.

b) What can we be conjectured regarding the expression of u_n in terms of n ?

c) Prove this conjecture by induction and to give the exact value of u_{2014}.

Exercise 12

Recall that the derivative of $g(x) = x$ is $g'(x) = 1$,
and that the derivative of a product is : $(uv)' = u'v + uv'$.

Given $n \in \mathbb{N}^*$, let f_n the function, defined for $x \in \mathbb{R}$, by : $f_n(x) = x^n$
Prove by induction that f_n is differentiable and that for any real number x : $f_n'(x) = nx^{n-1}$

Exercise 13

Let (u_n) be a sequence defined by : $\begin{cases} u_0 = 5 \\ u_{n+1} = \left(1 + \dfrac{2}{n+1}\right) u_n + \dfrac{6}{n+1} \end{cases}$

1) a) Calculate u_1 ; u_2 et u_3

   b) Let (d_n) be a sequence defined by : $d_n = u_{n+1} - u_n$.
   Write an algorithm to calculate u_n and d_{n-1} in terms of $n \geqslant 1$ then complete the following table :

n	0	1	2	3	4	5	6
u_n	5						
d_n							

   From these data conjecture the nature of the sequence (d_n).

2) Consider the arithmetic sequence (v_n) with a common difference of 8 and a first term of $v_0 = 16$.
   Justify that the sum of the n first terms of this sequence is equal to $4n^2 + 12n$.

3) Prove by induction that for all natural numbers n we have : $u_n = 4n^2 + 12n + 5$.

4) Validate the conjecture made in answer to question 1) b).

Limit of a sequence

In exercises 14, 15 and 16 determine the limit of the sequence (u_n) *using the theorems on the operations of limits.*

EXERCISE 14

1) $u_n = \dfrac{2n+5}{3n-2}$ 2) $u_n = \dfrac{n}{4} - 2 + \dfrac{2n}{n^2+5}$ 3) $u_n = \dfrac{-3n^2+2n+1}{2(n+1)^2}$

EXERCISE 15

1) $u_n = \dfrac{10n-3}{n^2-2}$ 2) $u_n = \dfrac{2n^2-1}{3n+2}$ 3) $u_n = \dfrac{3n^2-4}{n+1} - 3n$

EXERCISE 16

1) $u_n = \dfrac{\sqrt{n+2}}{n+2}$ 2) $u_n = \dfrac{n\sqrt{n}+n}{n-2}$

EXERCISE 17

Determine the limit of the following sequences using the comparison theorem :

a) $u_n = \dfrac{\cos(2n)}{\sqrt{n}}$, $n \in \mathbb{N}^*$ b) $v_n = n + 1 - \cos n$

EXERCISE 18

Let (u_n) be a sequence defined for $n \geqslant 1$ by : $u_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}$

a) Calculate u_1, u_2 and u_3

b) Given n, write an algorithm that calculates the general term u_n. Use the algorithm to determine the values of u_{10}, u_{20} and u_{50}. What can we conjecture about the limit of (u_n) ?

c) Prove that for $n \geqslant 1$: $\dfrac{n^2}{n^2+n} \leqslant u_n \leqslant \dfrac{n^2}{n^2+1}$

d) Deduce from the previous questions whether (u_n) is convergent. If so, what is the limit of the sequence ?

Limit of a geometric sequence

EXERCISE 19

Determine the limit of the sequence (u_n) such that : $u_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^n}$

EXERCISE 20

Let $(u_)$ be a sequence defined by : $u_0 = 3$ and $u_{n+1} = \dfrac{1}{3}u_n - 2$

Let (v_n) such that : $v_n = u_n + 3$.

1) a) Prove that the sequence (v_n) is geometric.
   b) Calculate v_n then u_n in terms of n
2) Let $S_n = v_0 + v_1 + \cdots + v_n$ and $S'_n = u_0 + u_1 + \cdots + u_n$
   a) Calculate S_n and S'_n in terms of n.
   b) Deduce limits of sequences (S_n) et (S'_n)

Exercise 21

Foreign centers in June 2013

Let (u_n) be a sequence defined by $u_1 = \dfrac{3}{2}$ et $u_{n+1} = \dfrac{nu_n + 1}{2(n+1)}$.

It defines an auxiliary sequence (v_n) by : for all natural numbers $n \geqslant 1$, $v_n = nu_n - 1$.

a) Show that the sequence (v_n) is geometric ;specify the common ratio and its first term.

b) Deduce, for all naturals numbers $n \geqslant 1$, we have : $u_n = \dfrac{1 + 0,5^n}{n}$.

c) Determine the limit of the sequence (u_n).

d) Justify that, for any natural number $n \geqslant 1$, we have : $u_{n+1} - u_n = -\dfrac{1 + 0,5^n(1 + 0,5n)}{n(n+1)}$.
   Can it be deduced that the sequence (u_n) is monotonic ?

Exercise 22

In the following cases, specify if the sequence (u_n) is bounded above, bounded below, bounded.

a) $u_n = \sin n$ b) $u_n = \dfrac{1}{1+n^2}$ c) $u_n = 2^n$

d) $u_n = n + \cos n$ e) $u_n = (-1)^n \times n^2$

Monotonic sequence

Exercise 23

Let (u_n) be a sequence defined by : $u_0 = 1$ and $u_{n+1} = u_n + 2n + 3$.

a) Study the monotonicity of the sequence (u_n).

b) What can be said about the convergence of the sequence (u_n) ?

Exercise 24

Answer true or false to the following proposals and justify your answer.

a) If a sequence is not bounded above then it tends to $+\infty$

b) If a sequence is increasing then it tends to $+\infty$

c) If a sequence tends to $+\infty$ then it is not bounded above.

d) If a sequence tends to $+\infty$ then it is increasing.

EXERCISE 25

Two ways to find the limit of a sequence

Let (u_n) be a sequence defined by : $u_0 = 0$ and $u_{n+1} = \dfrac{2u_n + 1}{u_n + 2}$

Part A : first method

1) a) Prove by induction that for all n, $0 \leqslant u_n < 1$

b) Verify that $u_{n+1} - u_n = \dfrac{1 - u_n^2}{u_n + 2}$ then show that the sequence (u_n) is increasing.

2) Deduce that the sequence (u_n) converges to a limit ℓ

3) Assuming that this limit ℓ verifies $f(\ell) = \ell$ with f defined on $[0; 1]$ by $f(x) = \dfrac{2x + 1}{x + 2}$

a) Determine the value of ℓ

b) Propose an algorithm to determine the value of N such that : $\forall n > N,\ |u_n - \ell| < 10^{-3}$. Enter this algorithm into your calculator and determine N.

Part B : second method

1) Let (v_n) be a sequence defined for all natural numbers n by : $\dfrac{u_n - 1}{u_n + 1}$

Prove that (v_n) is a geometric sequence. Specify the common ratio and the first term.

2) Express v_n, then u_n in terms of n.

3) Deduce that the sequence (u_n) is convergent and give its limit.

EXERCISE 26

North America in June 2013 - Extract

Consider the sequence (u_n) defined by $u_0 = 1$ and, for all natural numbers n,

$$u_{n+1} = \sqrt{2u_n}$$

1) Consider the following algorithm :

a) Give an approximate value to within 10^{-4} of the result printed by this algorithm when selecting $n = 3$.

b) What does this algorithm calculate ?

c) Complete the table below. We give the approximate values to within 10^{-4}

What conjectures can be made about the sequence (u_n) ?

```
Variables: n, i  natural numbers
           u : real number
Inputs and initialization
    Read n
    1 → u
Processing
    for i from 1 to n do
        √(2u) → u
    end
Sorties: Print u
```

n	1	5	10	15	20
Printed value					

2) a) Prove that for any natural number n, $0 < u_n \leqslant 2$.

b) Determine the monotonicity of the sequence (u_n).

c) Prove that the sequence (u_n) is convergent. The value of the limit is not required.

Exercise 27

For each of the following propositions, say whether it is true or false and justify the answer given.
Let (u_n) be the sequence defined for all $n \in \mathbb{N}^*$ by $u_n = (-1)^n$.

a) The sequence (u_n) is bounded.

b) The sequence (u_n) converges.

c) The sequence defined by $\dfrac{u_n}{n}$ converges.

d) Any decreasing sequence(v_n) having only strictly positive terms converges to 0.

Exercise 28

Metropolitan in June 2013

Let (u_n) be a sequence defined on $\mathbb{N}$ by : $\begin{cases} u_0 = 2 \\ u_{n+1} = \dfrac{2}{3}u_n + \dfrac{1}{3}n + 1 \end{cases}$

1) a) Calculate u_1, u_2, u_3 and u_4. Give the approximate values to within 10^{-2}.

   b) Make a conjecture on the monotonicity of this sequence.

2) a) Prove that for all natural numbers n : $u_n \leqslant n + 3$

   b) Prove that for all natural numbers n : $u_{n+1} - u_n = \dfrac{1}{3}(n + 3 - u_n)$

   c) Deduce from the previous question a proof of the conjecture.

3) Let (v_n) be the sequence defined on $\mathbb{N}$ by $v_n = u_n - n$.

   a) Prove that the sequence (v_n) is geometric sequence with a common ratio of $\dfrac{2}{3}$.

   b) Deduce that for all natural numbers n, $u_n = 2\left(\dfrac{2}{3}\right)^n + n$

   c) Determine the limit of the sequence (u_n).

4) For all natural numbers other than zero n, let : $S_n = \displaystyle\sum_{k=0}^{n} u_k = u_0 + u_1 + \cdots + u_n$ et $T_n = \dfrac{S_n}{n^2}$.

   a) Express S_n in terms of n.

   b) Determine the limit of the sequence (T_n).

Exercise 29

Lebanon in May 2013

Consider the sequence (v_n) defined by : $\begin{cases} v_0 = 1 \\ v_{n+1} = \dfrac{9}{6 - v_n} \end{cases}$

Part A

1) Write an algorithm that prints, for any given natural number n, all the terms of the sequence, from index 0 to index n.

2) Complete the following table for $n = 8$

n	0	1	2	3	4	5	6	7	8
u_n	1	1,800	2,143						

For $n = 100$, the last terms printed are :

2,967	2,968	2,968	2,968	2,969	2,969	2,969	2,970	2,970	2,970

What conjectures can be made regarding the sequence (v_n) ?

3) a) Prove by induction that, for all natural numbers n : $0 < v_n < 3$.

b) Prove, for all natural numbers n : $v_{n+1} - v_n = \dfrac{(3 - v_n)^2}{6 - v_n}$.

Is the sequence (v_n) monotonic ?

c) Prove that the sequence (v_n) converges.

Part B Finding the limit of the sequence (v_n)

Consider the sequence (w_n) defined by : $w_n = \dfrac{1}{v_n - 3}$

1) Prove that (w_n) is an arithmetic sequence with a common difference of $-\dfrac{1}{3}$

2) Deduce the expression of (w_n), and that of (v_n) in terms of n.

3) Determine the limit of the sequence (v_n).

Exercise 30

Antilles Guyana 2012 extract

Let (u_n) be the sequence defined by : $\begin{cases} u_1 = \dfrac{1}{2} \\ u_{n+1} = \dfrac{n+1}{2n} u_n \end{cases}$

1) Calculate u_2, u_3 et u_4.

2) a) Prove that, for all natural numbers n non zero, u_n is strictly positive.

b) Prove that the sequence (u_n) is decreasing.

c) What can be inferred for the sequence (u_n) ?

Exercise 31

Asia in June 2013

Part A

Consider the sequence (u_n) defined by : $u_0 = 2$ and $u_{n+1} = \dfrac{1 + 3u_n}{3 + u_n}$

We accept all terms of this sequence are defined and strictly positive.

1) Prove by induction that, for all natural numbers n, we have : $u_n > 1$.

2) a) Establish that for any natural number n, we have : $u_{n+1} - u_n = \dfrac{(1 - u_n)(1 + u_n)}{3 + u_n}$.

b) Determine the monotonicity of the sequence (u_n).
Deduce that the sequence (u_n) converges.

Part B

Consider the the sequence (u_n) defined by : $u_0 = 2$ and $u_{n+1} = \dfrac{1+0,5u_n}{0,5+u_n}$

We suppose all terms of this sequence are defined and strictly positive.

1) Consider the following algorithm : :

Copy and complete the following table, by running this algorithm for $n = 9$. The values of u are to be rounded to within 10^{-4}. Then conjecture the behavior of the sequence (u_n) as n approaches infinity.

Variables: n natural number
u real number
Inputs and initialization
Read n
$2 \to u$
Processing and outputs
for i from 1 to n **do**
$\dfrac{1+0.5u}{0.5+u} \to u$
Print u
end

i	1	2	3	4	5	6	7	8	9
u									

2) Consider the sequence (v_n) defined, for any natural numbers n, by : $v_n = \dfrac{u_n - 1}{u_n + 1}$.

a) Prove that the sequence (v_n) is geometric with a common ratio of $-\dfrac{1}{3}$.

b) Calculate v_0 then express v_n in terms of n.

3) a) Show that, for all natural numbers n, we have : $v_n \neq 1$.

b) Show that, for all natural numbers n, we have : $u_n = \dfrac{1+v_n}{1-v_n}$.

c) Determine the limit of the sequence (u_n).

Exercise 32

Antilles Guyana in June 2014

Let (u_n) be the sequence defined on $\mathbb{N}$ by : $\begin{cases} u_0 = 2 \\ u_{n+1} = \dfrac{1}{5}u_n + 3 \times 0,5^n \end{cases}$

1) a) Copy and, with your calculator, complete the value table of the sequence (u_n). Approximate the values to within 10^{-2} :

n	0	1	2	3	4	5	6	7	8
u_n	2								

b) According to this table, state a conjecture about the monotonicity of the sequence (u_n).

2) a) Prove by induction, that for all natural numbers n non zero : $u_n \geqslant \dfrac{15}{4} \times 0,5^n$.

b) Deduce that, for all natural numbers n non zero, $u_{n+1} - u_n \leqslant 0$.

c) Prove that the sequence (u_n) converges.

3) Let (v_n) be the sequence defined on $\mathbb{N}$ by $v_n = u_n - 10 \times 0,5^n$

a) Prove that the sequence (v_n) is a geometric sequence with a common ratio of $\dfrac{1}{5}$. Specify the first term of the sequence (v_n).

b) Deduce that, for all natural numbers n, $u_n = -8 \times \left(\dfrac{1}{5}\right)^n + 10 \times 0,5^n$

c) Determine the limit of the sequence (u_n)

4) Copy and complete the empty spaces in the following algorithm, in order to print the smallest value of n such that $u_n \leqslant 0,01$.

Then give the value found by the algorithm.

Variables: n : integer u : real number

Inputs and initialization

$0 \to n$

$2 \to u$

Processing

while …… **do**

…… $\to n$

…… $\to u$

end

Sorties: Print n