Continuity and differentiability of a function

Proofreading of English by Laurence Weinstock

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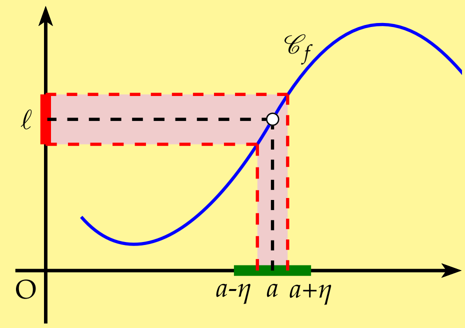
1 Continuity of a function

1.1 Finite limit at a finite point

Definition 1 : A function f has a finite limit ℓ at a if any open interval containing ℓ contains all values of $f(x)$ by choosing x close enough to a - i.e. all values of x in an interval of width 2η centered at a, $]a-\eta; a+\eta[$. The limit is denoted :

$$\lim_{x \to a} f(x) = \ell$$



Note : Sometimes, the function f does not have a limit at a, but does have a right- and left-hand limit such as the integer part function, denoted E in French and *int* in English (see below). For instance : $\lim_{x \to 2^-} E(x) = 1$ and $\lim_{x \to 2^+} E(x) = 2$

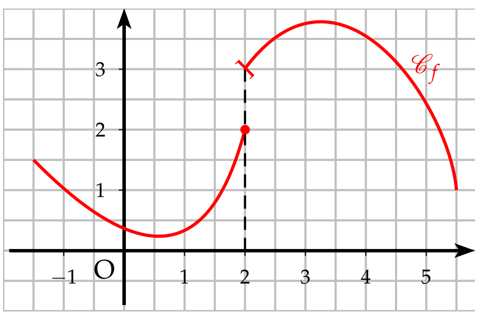
1.2 Continuity at a point

Definition 2 : Let f be a function defined on an interval I. Let a be an element of I. The function f is said to be **continuous** at a if, and only if :

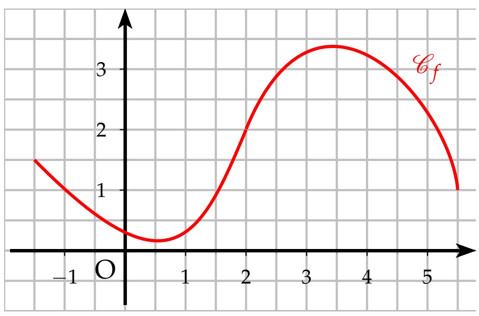
$$\lim_{x \to a} f(x) = f(a)$$

The function f is said to be **continuous on the interval I** if, and only if, f is continuous at each point in I.

Note : Less formally, a function is continuous when its graph is a single unbroken curve i.e. you could draw it without lifting your pen from the paper.



Function f discontinuous at 2
$\lim_{x \to 2^+} f(x) = 3 \neq f(2)$



Function f continuous on $[-1,5\ ;\ 5,5]$

On the left, the function f has a "jump" discontinuity. It is the case for example of the integer part function or more concretely of the function that represents the postal rates (abrupt rate change between letters below 20 g and for them between 20 g and 50 g in France).

There are other types of discontinuities. For example, the discontinuity at 0 of the function f defined by $f(x) = \sin\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.

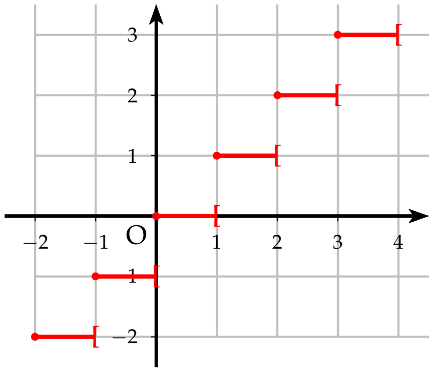
$\forall x \in \mathbb{R},\ \exists n \in \mathbb{Z},\ \ n \leqslant x < n+1$

The **integer part function** E is defined by : $E(x) = n \quad (int(x) = n$ in English)

$E(2,4) = 2\ ;\ \ E(5) = 5\ ;\ \ E(-1,3) = -2$

For the calculator Ti 82, (math) 5 : int.

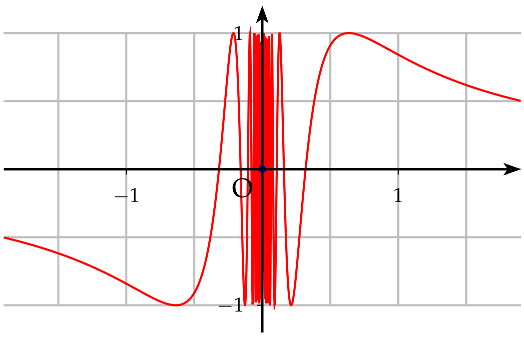
For any integer, there is a "jump" on the graph. So the integer part function is not continuous if x is an integer.



Let f be the function defined by :

$$\begin{cases} f(x) = \sin\frac{1}{x} & \text{if } x \neq 0 \\ f(0) = 0 \end{cases}$$

The function f is not continuous at 0 although no "jump" can be observed here. The function varies increasingly around 0 so that near 0, the function oscillates more and more. We cannot say that as x approaches 0 the function approaches 0.



1.3 Continuity of common functions

Property 1 : The following results can be proven

- All polynomials are continuous on $\mathbb{R}$.
- The simple rational function $x \mapsto \frac{1}{x}$ is continuous on $]-\infty; 0[$ and son $]0; +\infty[$
- The absolute value function $x \mapsto |x|$ is continuous on $\mathbb{R}$.
- The square root function $x \mapsto \sqrt{x}$ is continuous on $[0; +\infty[$
- The sine and cosine functions $x \mapsto \sin x$ and $x \mapsto \cos x$ are continuous on $\mathbb{R}$
- Generally speaking, all functions built by algebraic operation (addition, multiplication) or by composition from the above functions are continuous on their domain, in particular the rational functions.

1.4 Fixed point iteration theorem

Theorem 1 : Fixed point iteration theorem

Let (u_n) be a sequence defined by u_0 and a recurrence relation $u_{n+1} = f(u_n)$ converging to ℓ.
If the associated function f is continuous at ℓ, then the limit of the sequence, ℓ, is a solution of the equation $f(x) = x$. It is said that ℓ is a fixed point of f

Proof :

We know that the sequence (u_n) converges to ℓ so : $\lim_{n \to +\infty} u_n = \ell$

Moreover, the function f is continuous at ℓ so : $\lim_{x \to \ell} f(x) = f(\ell)$

By composition, we deduce that : $\lim_{n \to +\infty} f(u_n) = f(\ell) \Leftrightarrow \lim_{n \to +\infty} u_{n+1} = f(\ell)$

but $\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} u_{n+1}$ so $\ell = f(\ell)$

Example : Let (u_n) be a sequence defined by

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \sqrt{3u_n + 4} \end{cases}$$

In chapter 2, we proved by induction that (u_n) is positive, increasing and bounded above by 4, according to the theorem of monotonic sequences, (u_n) is convergent to ℓ. The function $x \mapsto \sqrt{3x+4}$ is continuous on $[0;4]$, so ℓ is solution to the equation $f(x) = x$.

$$\begin{aligned} \sqrt{3x+4} &= x \quad \text{by squaring} \\ 3x + 4 &= x^2 \\ x^2 - 3x - 4 &= 0 \end{aligned}$$

This equation has two solutions -1 and 4. As the sequence (u_n) is positive then, according to the fixed point theorem, the sequence (u_n) converges to 4.

1.5 Continuity and differentiability

Theorem 2 : Differentiability implies continuity

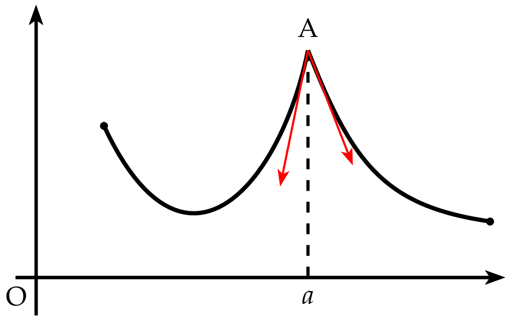
- If f is differentiable at a point a then the function f is continuous at a.
- If f is differentiable on an interval I then the function f is continuous on I.

⚠ The converse of this theorem is false

Note : The converse of this theorem is false. A graph can have no jump discontinuity but not have a tangent line at a finite point as in the following example :

This function therefore is continuous at a (no jump), but is not differentiable at a. The graph has no tangent line at a.

The graph is said have a corner or **cusp** at the point A



The absolute value function $x \mapsto |x|$ is continuous but not differentiable at 0.

1.6 Continuity and equation

Theorem 3 : Intermediate value theorem (IVT)

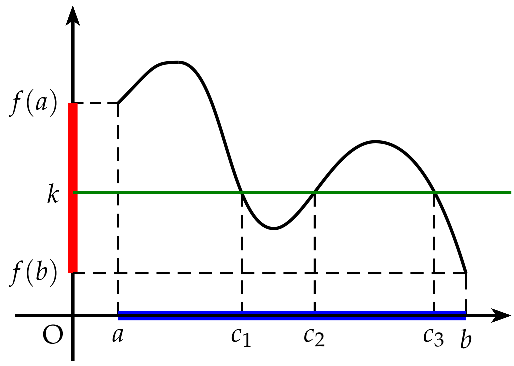
Let f be a **continuous** function on an interval $\text{I} = [a, b]$.
For any real number k between $f(a)$ and $f(b)$, there must be at least one value $c \in \text{I}$ such that $f(c) = k$.

Note : The proof of this theorem is not on the syllabus.

This theorem follows from the fact that the **image of an interval** of a continuous function over an interval of $\mathbb{R}$ **is itself an interval** of $\mathbb{R}$

Consider the following graph : k is between $f(a)$ and $f(b)$. Note the wording of the theorem

"*at least one value*". This means we could have more. Here for example, we have 3 points where $f(x) = k$. The equation $f(x) = k$ therefore has 3 solutions : c_1, c_2 and c_3.



Theorem 4 : IVT with continuous one-to-one functions

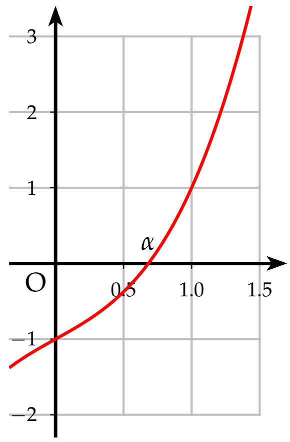
Let f be a **continuous and strictly monotonic** function on $I = [a, b]$.
For any k between $f(a)$ and $f(b)$, the equation $f(x) = k$ has a unique solution in the interval $\text{I} = [a, b]$

Proof : The existence is proven by the precedent theorem, and the uniqueness by the monotonicity of the function.

Note :

- This theorem is generalized with an open interval $I =]a, b[$. k must be between $\lim_{x \to a} f(x)$ and $\lim_{x \to b} f(x)$
- When $k = 0$, we then have to show $f(a) \times f(b) < 0$.
- Sometimes, in French, the theorem is called the "*bijective theorem*" because the function is bijective from I to $f(\text{I})$.

Example : Let f be a function defined on $\mathbb{R}$ by : $f(x) = x^3 + x - 1$. Show that the equation $f(x) = 0$ has only one solution on $\mathbb{R}$. Give an approximation to within one unit. Then find , with an algorithm, an approximate value within 10^{-6} of the solution.



The function f is a **continuous** function on $\mathbb{R}$ because f is a polynomial.

The function f is the sum of increasing functions $x \mapsto x^3$ and $x \mapsto x-1$, so f is **strictly increasing** on $\mathbb{R}$.

We have $f(0)=-1$ and $f(1)=1 \Rightarrow f(0) \times f(1) < 0$

hence by the intermediate value theorem, the function f have an only one solution $\alpha \in [0,1]$ such that $f(\alpha) = 0$.

Algorithm : An algorithm, using the **dichotomy** principle (the interval is divided in two and the operation is repeated) one can find a value of α rounded to within 10^{-6}.
Consider :

- A and B endpoints of the interval.
- P the accuracy (natural number).
- N the number of iterations.

Input : $A = 0$, $B = 1$, $P = 6$ and $f(x) = x^3 + x - 1$

We find : $A = 0{,}682\,327$, $B = 0{,}682\,328$ and $N = 20$.

20 iterations are required to have an accuracy of 10^{-6}

```
Variables: A, B, C  real numbers
           P, N  integers  f  function
Inputs and initialization
    Read A, B, P
    0 → N
Processing
    while B − A > 10^(−P) do
        (A + B)/2 → C
        if f(A) × f(C) > 0 (*) then
            C → A
        else
            C → B
        end
        N + 1 → N
    end
Output   : Print : A, B, N
```

⚠ This algorithm is made for $k = 0$, but we can change the algorithm for any real numbers k :

- ask to read K and then make the following change in the algorithm : $(f(A)-K)\times(f(C)-K) > 0$
- or input the function g instead of f such that : $g(x) = f(x) - k$

2 Differentiability

2.1 Definition

Definition 3 : Let f be a function defined on an open interval I and a a point of I. The function f is said to be differentiable at a if and only if the rate of change of the function f at a has a finite limit ℓ at a, i.e. :

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \ell$$

ℓ is called the derived number of f at a and is denoted $f'(a)$

When the function f is differentiable on an interval I, the derivative function, called f', which to x of I relates the derived number $f'(x)$.

Note :

- If the function f is differentiable at a then the function f is continuous at a
- Physicists express a variation with the symbol Δ ; they therefore write :
  $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$.
  For an infinitesimally small variation, they write $\mathrm{d}x$ and $\mathrm{d}y$. The notation (Leibniz's notation) for the derivative is :

$$f' = \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad f'(a) = \frac{\mathrm{d}y}{\mathrm{d}x}(a)$$

Example : Let f be a piecewise function defined by :

$$\begin{cases} f(x) = x^2 - 2x - 2 & \text{if } x \leqslant 1 \\ f(x) = \dfrac{x-4}{x} & \text{if } x > 1 \end{cases}$$

Let us study the continuity and differentiability of this function at 1.

- **Continuity at 1**. Left-hand continuity at 1 is not a problem, because a polynomial is continuous on $]-\infty\,;1]$. For the right-hand continuity :

$$\lim_{x \to 1^+} \frac{x-4}{x} = -3 \quad \text{and} \quad f(1) = 1^2 - 2 \times 1 - 2 = -3$$

  so : $\displaystyle\lim_{x \to 1^+} \frac{x-4}{x} = f(1)$ the function f is continuous at 1

- **Differentiability at 1**. Left-hand differentiability at 1 is not a problem because a polynomial is differentiable on $]-\infty;1]$.

  if $x \leqslant 1$, we have $f'(x) = 2x - 2$ so $f'_-(1) = 0$

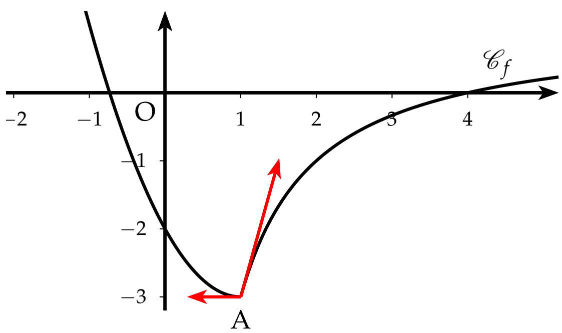
  As for right-hand differentiability, we have to revert to the definition. We can then carry out the following calculation :

$$\frac{f(1+h) - f(1)}{h} = \frac{\dfrac{1+h-4}{1+h} + 3}{h} = \frac{4h}{h(1+h)} = \frac{4}{1+h}$$

  So : $\displaystyle\lim_{h \to 0^-} \frac{4}{1+h} = 4$ then $f'_+(1) = 4$

  As $f'_-(1) \neq f'_+(1)$ the function f is not differentiable at 1.

  Graphically the graph $\mathscr{C}_f$ is a single unbroken curve and has a cusp at the point A.



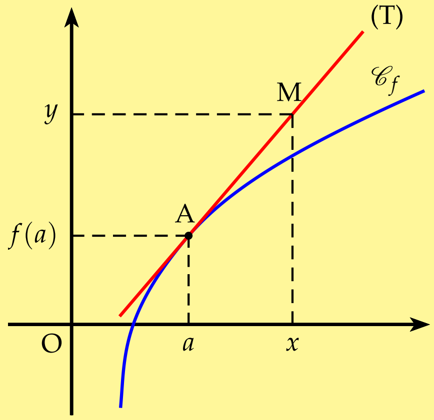
2.2 Interpretations

2.2.1 Graphical interpretation

Theorem 5 :

When f is differentiable at a, the graph $\mathscr{C}_f$ of the function f has, at the point $A(a, f(a))$ a tangent line with a linear coefficient $f'(a)$ whose the equation is :

$$\text{(T)}: \quad y = f'(a)(x-a) + f(a)$$



Note : The derived number represents the slope of the tangent line at a point. If the function f is left- and right-hand differentiable but not differentiable at a, the graph $\mathscr{C}_f$ has two one-sided tangent lines.

2.2.2 Numerical interpretation

Theorem 6 : When a function f is differentiable at a, a good linear approximation, when $a+h$ approaches a is :

$$f(a+h) \approx f(a) + hf'(a)$$

Example : Determine a linear approximation of $\sqrt{4,03}$.

Consider $f(x) = \sqrt{x}, \quad a = 4 \quad$ and $\quad h = 0,03$. We calculate the derivative at 4.

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{so} \quad f'(4) = \frac{1}{4}$$

$$\text{then} \quad f(4,03) \approx f(4) + 0,03 \times \frac{1}{4} \approx 2,0075$$

Hence : $\sqrt{4,03} \approx 2,0075$ which is comparable to the calculator value 2,007 486. The accuracy is 10^{-4}.

2.2.3 Kinematic interpretation

If $x(t)$ is an equation of motion, then $x'(t)$ represents the instantaneous velocity at the time t. Same, if $v(t)$ is the instantaneous velocity at the time t, then $v'(t)$ represents the acceleration at the time t.

With physicist notation, the instantaneous velocity v and the acceleration a is denoted :

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

2.3 Monotonicity and sign of the derivative

Theorem 7 : Let f be a function differentiable on an interval I.

- If the derivative f' is *zero*, then the function f is *constant*.
- If the derivative f' is *strictly positive* (except in a few points of I where is zero), then the function f is *strictly increasing* on I.
- If the derivative is *strictly negative* (except in a few points of I where is zero), then the function f is *strictly decreasing* on I.

Note : The monotonicity of a differentiable function is given by the sign of the derivative.

Example : Study the monotonicity of the function f defined on $\mathbb{R}$ by :

$$f(x) = x^3 - 6x^2 + 1$$

f is differentiable on $\mathbb{R}$ and :

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$

Then

- $f'(x) = 0 \quad \Leftrightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 4$
- f' is positive outside the roots and negative inside.

We can sum up the monotonicity of the function with the following table :

x	$-\infty$		0		4		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	↗	1	↘	-31	↗	$+\infty$

2.4 Derivative and relative extrema

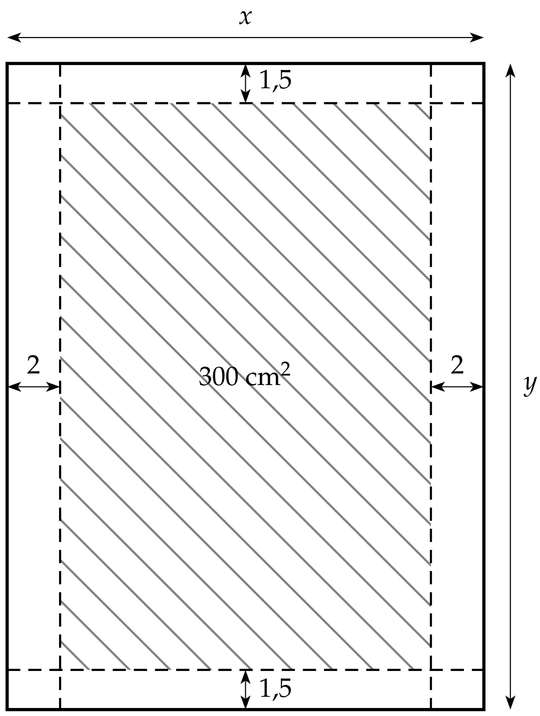
Theorem 8 : Let f be a differentiable function on an open interval I and a a point of I.

- If f has a relative extremum at a then $f'(a) = 0$.
- If $f'(a) = 0$ and if f' changes sign at a then the function f has a relative extremum at a.

Note : Relative extrema are sought among the zeros of the derivative, but if $f'(a) = 0$, a is not necessarily a relative extremum (counterexample $f(x) = x^3$ at $a = 0$).

Consequently Optimization problems include determining a differentiable function and determining relative extrema.

Example : Problem of the publisher.
A publisher wishes to produce a book with the following constraints : on each page the printed text must be contained within a rectangle of 300 cm^2, the margins should measure 1.5 cm on the horizontal edges and 2 cm on the vertical edges .
What should be the size of a page that paper consumption is minimal ?
Consider x and y the horizontal and vertical dimensions and S the total area of the sheet. We seek to express y and S in terms of x. As paper consumption is given by the surface, the minimum of S will be determined according to the values of x



The printed area : $(x-4)(y-3) = 300 \quad \Leftrightarrow \quad y = \dfrac{300}{x-4} + 3$

The total area : $S(x) = xy = \dfrac{300x}{x-4} + 3x = 3\left(\dfrac{100x}{x-4} + x\right)$

We derive S : $S'(x) = 3\left(\dfrac{100(x-4) - 100x}{(x-4)^2} + 1\right) = 3\left(\dfrac{-400 + x^2 - 8x + 16}{(x-4)^2}\right)$

$$S'(x) = \frac{3(x^2 - 8x + 384)}{(x-4)^2}$$

Zero of S' : $S'(x) = 0 \quad \Leftrightarrow \quad x^2 - 8x + 384 = 0$

hence $\Delta = 8^2 + 4 \times 384 = 1600 = 40^2$

The positive root is : $x = \dfrac{8+40}{2} = 24$

x	4		24		$+\infty$
$S'(x)$		$-$	0	$+$	
$S(x)$		↘		↗	

The total area is at a minimum when $x = 24$, we then deduce $y = \dfrac{300}{24-4} + 3 = 18$

The dimensions of the sheet that makes minimal paper consumption is 24×18 cm

2.5 Derivatives of common functions and derivative rules

2.5.1 Derivative of common functions

Here is the table of the derivatives of common functions and their domain

Function	Dom(f)	Derivative	Dom(f')
$f(x) = k$	$\mathbb{R}$	$f'(x) = 0$	$\mathbb{R}$
$f(x) = x$	$\mathbb{R}$	$f'(x) = 1$	$\mathbb{R}$
$f(x) = x^n \quad n \in \mathbb{N}^*$	$\mathbb{R}$	$f'(x) = nx^{n-1}$	$\mathbb{R}$
$f(x) = \dfrac{1}{x}$	$\mathbb{R}^*$	$f'(x) = -\dfrac{1}{x^2}$	$]-\infty;0[$ or $]0;+\infty[$
$f(x) = \dfrac{1}{x^n} \quad n \in \mathbb{N}^*$	$\mathbb{R}^*$	$f'(x) = -\dfrac{n}{x^{n+1}}$	$]-\infty;0[$ or $]0;+\infty[$
$f(x) = \sqrt{x}$	$[0;+\infty[$	$f'(x) = \dfrac{1}{2\sqrt{x}}$	$]0;+\infty[$
$f(x) = \sin x$	$\mathbb{R}$	$f'(x) = \cos x$	$\mathbb{R}$
$f(x) = \cos x$	$\mathbb{R}$	$f'(x) = -\sin x$	$\mathbb{R}$

2.5.2 Derivative rules

Sum rule	$(u+v)' = u' + v'$
Multiplication by constant	$(ku)' = ku'$
Product rule	$(uv)' = u'v + uv'$
Reciprocal rule	$\left(\dfrac{1}{u}\right)' = -\dfrac{u'}{u^2}$
Quotient rule	$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$
Power rule	$(u^n)' = nu'u^{n-1}$
Square root rule	$(\sqrt{u})' = \dfrac{u'}{2\sqrt{u}}$
Chain rule	$[f(ax+b)]' = a \times f'(ax+b)$

Note : The last three are new derivation rules in the 12th grade

2.5.3 Examples

Determine the derivatives of the following functions :

a) $f(x) = (3x-5)^4$ b) $g(x) = \sqrt{x^2+x+1}$ c) $h(x) = \sin(2x+1)$

These three function are differentiable on $\mathbb{R}$ because they are the sum, product and composite of differentiable functions. We then have :

a) $f'(x) = 4 \times 3(3x-5)^3 = 12(3x-5)^3$

b) $g'(x) = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$

c) $h'(x) = 2\cos(2x+1)$