Complex numbers

Proofreading of English by Laurence Weinstock

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1 Introduction

1.1 A historical problem

At the end of the 16th century, there was a great deal of interest in solving cubic equations. It was quickly shown that by changing variables any cubic equation can be written in the form

$$x^3 + px + q = 0$$

This equation has at least one real root, which can be expressed in the form :

$$x_0 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

An Italian mathematician of the time, Bombelli, was particularly interested in the following equation :

$$x^3 - 15x - 4 = 0$$

A solution to which can be found as follows : $p = -15$ and $q = -4$

$$\begin{aligned} x_0 &= \sqrt[3]{2 - \sqrt{4 - 125}} + \sqrt[3]{2 + \sqrt{4 - 125}} \\ &= \sqrt[3]{2 - \sqrt{-121}} + \sqrt[3]{2 + \sqrt{-121}} \\ &= \sqrt[3]{2 - 11\sqrt{-1}} + \sqrt[3]{2 + 11\sqrt{-1}} \end{aligned}$$

However, the square root $\sqrt{-1}$ was problematic.

But Bombelli noticed that by using the expression $(\sqrt{-1})^2 = -1$, he could carry out the following expansion [1]

$$\begin{aligned} (2 - \sqrt{-1})^3 &= 2^3 - 3(2)^2\sqrt{-1} + 3(2)(\sqrt{-1})^2 - (\sqrt{-1})^3 \\ &= 8 - 12\sqrt{-1} + 6(-1) - (-1)\sqrt{-1} \\ &= 2 - 11\sqrt{-1} \end{aligned}$$

$$\begin{aligned} (2 + \sqrt{-1})^3 &= 2^3 + 3(2)^2\sqrt{-1} + 3(2)(\sqrt{-1})^2 + (\sqrt{-1})^3 \\ &= 8 + 12\sqrt{-1} + 6(-1) + (-1)\sqrt{-1} \\ &= 2 + 11\sqrt{-1} \qquad \text{then} \end{aligned}$$

$$x_0 = 2 - \sqrt{-1} + 2 + \sqrt{-1} = 4$$

And indeed , 4 is a solution to the equation.

$$4^3 - 15 \times 4 - 4 = 64 - 60 - 4 = 0$$

Conclusion : $\sqrt{-1}$ does not exist, but allows you to find the solution to an equation via an intermediate calculation. Complex numbers were born ! !

- In the 17th century these numbers became intermediaries for common calculations but were not considered as numbers in their own right.

1. Remember that : $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

- In the 18th century it was shown that these numbers can be put into the form $a + b\sqrt{-1}$.
  Euler then proposed to denote them thus $\sqrt{-1} = i$. i being called "imaginary".
- In the 19th century Gauss showed that such numbers can be represented graphically. They then finally received the status of numbers.

1.2 Creating a new set of numbers

The discovery of a new set of numbers is quite common in mathematics. Let us recall the solutions to the following equations.

- Resolution in $\mathbb{N}$ of the equation $x + 7 = 6$.

  This equation has no solution, but by creating the negative integers, we find that $x = -1$
- Resolution in $\mathbb{Z}$ of the equation $3x = 1$.

  This equation has no solution, but by creating rational numbers, we find that $x = \dfrac{1}{3}$.
- Resolution in $\mathbb{Q}$ of the equation $x^2 = 2$.

  This equation has no solution, but by creating real numbers, we find that $x = \sqrt{2}$ or $x = -\sqrt{2}$.
- Resolution in $\mathbb{R}$ of the equation $x^2 + 1 = 0$.

  This equation has no solution. So by creating a set of numbers called $\mathbb{C}$ (for complex) whose main characteristic by comparison with the real numbers is the addition of the number i such that $i^2 = -1$, the following solutions can be found : $x = i$ and $x = -i$

The natural approach is therefore to seek a greater set of numbers that contains the former, and which possesses the same properties and which can be represented graphically.

2 The construction of complex numbers

2.1 Definition

Definition 1 : The set of complex numbers $\mathbb{C}$, is the set of numbers z of the form :

$$z = a + ib \quad \text{with}(a, b) \in \mathbb{R}^2 \quad \text{and} \quad i^2 = -1$$

The real number a is called the real part of z denoted : $Re(z)$
The real number b is called the imaginary part of z denoted : $Im(z)$.
This form $z = a + ib$ is called the Cartesian form.

Note :

1) All real numbers are included in $\mathbb{C}$ (let $b = 0$).

2) If $a = 0$, z is purely imaginary

2.2 Representation of complex numbers

Theorem 1 : A point $M(a\,;\,b)$ can be related to any complex number $z = a + ib$, in an orthonormal plane$(O, \overrightarrow{u}, \overrightarrow{v})$
z is called the **affix** of point M, and is written $M(z)$.

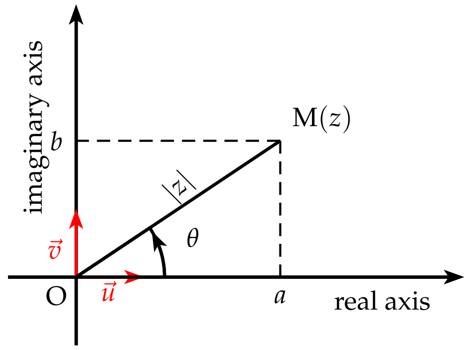
Note : This function is bijective. Conversely, a complex number $z = x + iy$ can also be related to any point $M(x\,;\,y)$ of a plane with an orthonormal basis.

Conclusion : The complex number $z = a + ib$ can be represented graphically.

The modulus of z is the distance OM, i.e. the quantity, denoted $|z|$ such that :

$$|z| = \sqrt{a^2 + b^2}$$

If $z \in \mathbb{R}$, then $z = a$ and $|z| = \sqrt{a^2} = |a|$
That is to say, the modulus is the absolute value of the real number (it has the same nature, so therefore the same notation).



And for $z \neq 0$, the argument of z, denoted $\arg(z)$, is the angle θ, $(\vec{u}; \overrightarrow{OM})$ such that :

$$\begin{cases} \cos\theta = \dfrac{a}{|z|} \\ \sin\theta = \dfrac{b}{|z|} \end{cases} \quad \text{with} \quad \theta = \arg(z) \quad [2\pi]$$

Examples :

1) Determine the modulus and an argument of the following complex numbers :
$z_1 = 1 + i \quad , \quad z_2 = 1 - \sqrt{3}i \quad , \quad z_3 = -4 + 3i$

$|z_1| = \sqrt{1+1} = \sqrt{2}$
$$\begin{cases} \cos\theta_1 = \dfrac{1}{\sqrt{2}} \\ \sin\theta_1 = \dfrac{1}{\sqrt{2}} \end{cases}$$
$\theta_1 = \dfrac{\pi}{4}$

$|z_2| = \sqrt{1+3} = 2$
$$\begin{cases} \cos\theta_2 = \dfrac{1}{2} \\ \sin\theta_2 = -\dfrac{\sqrt{3}}{2} \end{cases}$$
$\theta_2 = -\dfrac{\pi}{3}$

$|z_3| = \sqrt{16+9} = 5$
$$\begin{cases} \cos\theta_3 = -\dfrac{4}{5} \\ \sin\theta_3 = \dfrac{3}{5} \end{cases}$$
$\theta_3 = \arccos -\dfrac{4}{5} \simeq 143°$

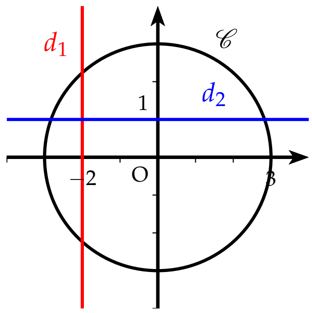
2) Determine the set of points M whose affix z satisfies each of the following equalities :

a) $|z| = 3$ b) $\text{Re}(z) = -2$ c) $\text{Im}(z) = 1$

a) $|z| = 3$: circle $\mathscr{C}$ centered at O and of radius 3

b) $\text{Re}(z) = -2$: A line d_1 parallel to the y-axis, with an x-coordinate of -2

c) $\text{Im}(z) = 1$: A line d_2 parallel to the x-axis, with a y-coordinate of 1



2.3 Operations with complex numbers

Two operations can be defined in the set of complex numbers :

- **Addition (+)** :

$$\text{if} \quad z = a + ib \quad \text{and} \quad z' = a' + ib' \quad \text{then} \quad z + z' = (a + a') + i(b + b')$$

- **Multiplication (×)** :

$$\text{if} \quad z = a + ib \quad \text{and} \quad z' = a' + ib' \quad \text{then} \quad z \times z' = (aa' - bb') + i(ab' + a'b)$$

The set of complex numbers $\mathbb{C}$ under the laws of addition and multiplication is a commutative field. It has all the properties of the two laws in the set of real numbers $\mathbb{R}$, i.e. : commutativity and associativity for addition and multiplication, the distribution of multiplication over addition, ...

A complex number is equal to zero, if and only if its real and imaginary parts are equal to zero :

$$a + ib = 0 \quad \Leftrightarrow \quad a = 0 \quad \text{and} \quad b = 0$$

Examples : Consider the following :

$$z_1 = 4 + 7i - (2 + 4i) = 4 + 7i - 2 - 4i = 2 + 3i$$
$$z_2 = (2 + i)(3 - 2i) = 6 - 4i + 3i + 2 = 8 - i$$
$$z_3 = (4 - 3i)^2 = 16 - 24i - 9 = 7 - 24i$$

Note : **Comparison of two complex numbers** : it is possible to define an order in $\mathbb{C}$ which is a continuation of the order in $\mathbb{R}$. We can simply compare the real parts and if they are equal we then compare the imaginary parts. Denoting "$\preceq$" such an order, we would have :

$$a + ib \preceq c + id \quad \Leftrightarrow \quad a < c \quad \text{or} \quad a = c \text{ and } b \leqslant d$$

What follows is : $2 + 5i \preceq 3 - 7i$ and $-1 - i \preceq -1 + 2i$

However, this order is not a comprehensive order because it is not compatible with multiplication :

$$\text{according to this order} : 0 \preceq i \quad \text{but by multiplying by } i \quad 0 \not\preceq -1$$

So the idea of inequalities in $\mathbb{C}$ was finally abandoned !

2.4 The complex conjugate

2.4.1 Definition

Definition 2 : Consider a complex number z with the Cartesian form : $z = a + ib$. The conjugate of z, is the number, denoted $\bar{z}$, such that :

$$\bar{z} = a - ib$$

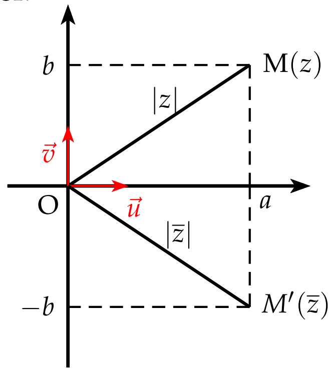
Property : The product of a complex number and its conjugate is :

$$\boxed{z\bar{z} = |z|^2 = a^2 + b^2} \quad \text{indeed : } (a+ib)(a-ib) = a^2 - iab + iab + b^2$$

This way we can make a denominator a real number.

Geometric interpretation

The point $M'(\bar{z})$ is the symmetrical point $M(z)$ relative to the x-axis.



2.4.2 Applications

1) Find the Cartesian form of the following complex number : $z = \dfrac{2-i}{3+2i}$

Multiplying the top and bottom of the fraction by the complex conjugate of the denominator :

$$z = \frac{(2-i)(3-2i)}{(3+2i)(3-2i)} = \frac{6-4i-3i-2}{9+4} = \frac{4-7i}{13} = \frac{4}{13} - \frac{7}{13}i$$

2) Solve the following equation : $z = (2-i)z + 3$

$$z = (2-i)z + 3$$
$$z - (2-i)z = 3$$
$$z(1-2+i) = 3$$
$$z = \frac{3}{-1+i} = \frac{-3}{1-i}$$
$$z = \frac{-3(1+i)}{(1-i)(1+i)}$$
$$z = -\frac{3}{2} - \frac{3}{2}i$$

2.4.3 Properties

Property 1 : If z is a complex number and $\bar{z}$ its conjugate, then we have the following properties :

$z + \bar{z} = 2\text{Re}(z)$ and "z is a purely imaginary" equivalent to : $z + \bar{z} = 0$

$z - \bar{z} = 2i\,\text{Im}(z)$ and "z is a real number" equivalent to : $z = \bar{z}$

Law 1 : For all complex numbers z and z' :

$$\overline{z+z'} = \overline{z} + \overline{z'} \quad , \quad \overline{z \times z'} = \overline{z} \times \overline{z'}$$

$$\text{with } z' \neq 0 \quad \overline{\left(\frac{z}{z'}\right)} = \frac{\overline{z}}{\overline{z'}} \quad , \quad \overline{z^n} = (\overline{z})^n \quad n \in \mathbb{N}^*$$

Examples :

1) Find the Cartesian form of the conjugate $\overline{z}$ of the following complex number : $z = \dfrac{3-i}{1+i}$

$$\overline{z} = \overline{\left(\frac{3-i}{1+i}\right)} = \frac{\overline{3-i}}{\overline{1+i}} = \frac{3+i}{1-i} = \frac{(3+i)(1+i)}{1+1} = \frac{3+3i+i-1}{2} = 1+2i$$

2) M is a point in the complex plane with the affix $z = x+iy$, x and y are real numbers. Let the number : $Z = \dfrac{5z-2}{z-1}$ be related to all complex numbers z, $z \neq 1$

a) Express $Z + \overline{Z}$ in terms of z and $\overline{z}$.

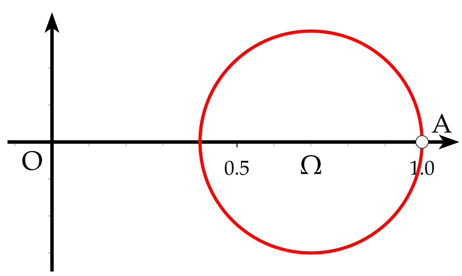
b) Prove that "Z is purely imaginary" is equivalent to "M is a point on a circle missing a point".

a)
$$\begin{aligned} Z + \overline{Z} &= \frac{5z-2}{z-1} + \overline{\left(\frac{5z-2}{z-1}\right)} \\ &= \frac{5z-2}{z-1} + \frac{5\overline{z}-2}{\overline{z}-1} \\ &= \frac{(5z-2)(\overline{z}-1) + (5\overline{z}-2)(z-1)}{(z-1)(\overline{z}-1)} \\ &= \frac{5z\overline{z} - 5z - 2\overline{z} + 2 + 5z\overline{z} - 5\overline{z} - 2z + 2}{(z-1)(\overline{z}-1)} \\ &= \frac{10z\overline{z} - 7(z+\overline{z}) + 4}{(z-1)(\overline{z}-1)} \end{aligned}$$

b) If Z is purely imaginary then $Z + \overline{Z} = 0$. So therefore :

$$\begin{aligned} 10z\overline{z} - 7(z+\overline{z}) + 4 &= 0 \\ 10|z|^2 - 14\text{Re}(z) + 4 &= 0 \\ 10(x^2+y^2) - 14x + 4 &= 0 \\ x^2 + y^2 - \frac{7}{5} + \frac{2}{5} &= 0 \\ \left(x - \frac{7}{10}\right)^2 - \frac{49}{100} + y^2 + \frac{2}{5} &= 0 \\ \left(x - \frac{7}{10}\right)^2 + y^2 - \frac{9}{100} &= 0 \end{aligned}$$

$$\left(x - \frac{7}{10}\right)^2 + y^2 = \left(\frac{3}{10}\right)^2$$



It can therefore be deduced that the set of points $M(z)$ is the circle centered at $\Omega\left(\dfrac{7}{10}\right)$ and of radius $\dfrac{3}{10}$ missing the point A(1).

3 Quadratic equations

3.1 Resolution

Complex numbers were created so that quadratic equations always have roots.

Theorem 2 : Any quadratic equation in $\mathbb{C}$ always has either two distinct solutions or one double solution. If the equation has real coefficients, i.e.

$$az^2 + bz + c = 0 \qquad \text{with } a \in \mathbb{R}^*,\, b \in \mathbb{R} \text{ and } c \in \mathbb{R}$$

then it has solutions in $\mathbb{C}$.

1) If $\Delta > 0$, two real solutions : $z_1 = \dfrac{-b+\sqrt{\Delta}}{2a}$ and $z_2 = \dfrac{-b-\sqrt{\Delta}}{2a}$

2) If $\Delta = 0$, a double real solution : $z_0 = -\dfrac{b}{2a}$

3) If $\Delta < 0$, two conjugate complex solutions with $\Delta = i^2|\Delta|$

$$z_1 = \frac{-b+i\sqrt{|\Delta|}}{2a} \quad \text{and} \quad z_2 = \frac{-b-i\sqrt{|\Delta|}}{2a}$$

Example : Solve $z^2 - 2z + 2 = 0$

$\Delta = 4 - 8 = -4 = (2i)^2$.
$\Delta < 0$ so there are two conjugate complex solutions :

$$z_1 = \frac{2+2i}{2} = 1 + i$$
$$z_2 = \frac{2-2i}{2} = 1 - i$$

We can write an algorithm to calculate the roots of : $Ax^2 + Bx + C$

```
Variables: A, B, C, D, X, Y   real numbers
Inputs and initialization
    Input A, B, C
    B^2 − 4AC → D
Processing
    if D ⩾ 0 then
        (−B + √D)/(2A) → X
        (−B − √D)/(2A) → Y
    else
        (−B + i√|D|)/(2A) → X
        (−B − i√|D|)/(2A) → Y
    end
Output   : Print X, Y
```

3.2 Applications to higher degree equations

Theorem 3 : Any polynomial of degree n in $\mathbb{C}$ has n distinct or multiple roots. If a is a root then the polynomial can be factorized by $(z - a)$

Example : Consider following equation in $\mathbb{C}$: $z^3-(4+i)z^2+(13+4i)z-13i=0$

1) Prove that i is a solution to the equation
2) Determine the real numbers a, b and c such that :
   $z^3-(4+i)z^2+(13+4i)z-13i=(z-i)(az^2+bz+c)$.
3) Solve the equation.

1) Find all check that i is indeed a solution to the equation :
   $i^3-(4+i)i^2+(13+4i)i-13i=-i+4+i+13i-4-13i=0$
   The quantity $(z-i)$ can therefore be factored out, i being a solution to the equation.
2) The coefficients can be identified by expanding the expression into its initial form :

$$\begin{aligned}(z-i)(az^2+bz+c)&=az^3+bz^2+cz-iaz^2-ibz-ic\\&=az^3+(b-ia)z^2+(c-ib)z-ic\end{aligned}$$

   The following system of equations is obtained :

$$\begin{cases}a=1\\b-ia=-4-i\\c-ib=13+4i\\-ic=-13i\end{cases}\Leftrightarrow\begin{cases}a=1\\b=-4\\c=13\end{cases}$$

3) The equation becomes : $(z-i)(z^2-4z+13)=0$
   So therefore $z=i$ or $z^2-4z+13=0$.
   $\Delta=16-52=-36=(6i)^2$
   Two conjugate complex solutions are obtained :

$$z_1=\frac{4+6i}{2}=2+3i \quad\text{or}\quad z_2=\frac{4-6i}{2}=2-3i$$

   Conclusion : $S=\{i\,;\,2-3i\,;\,2+3i\}$

4 Polar and exponential form

4.1 Polar form

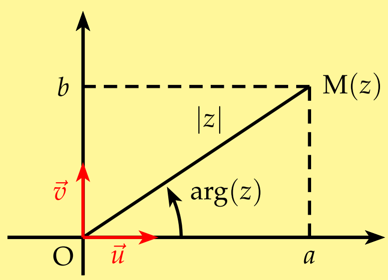
4.1.1 Definition

Definition 3 : The polar form of a complex number z $(z\neq 0)$ whose Cartesian form is $a+ib$, is :

$$z=r(\cos\theta+i\sin\theta)$$

with

$$r=|z| \quad\text{and}\quad \theta=\arg(z)\quad[2\pi]$$



Note : The polar form is related to the polar coordinates of a point.

Examples :

1) Find the polar form of $z = 1 - i$
   First determine the modulus : $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
   We determine an argument : $\cos\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ and $\sin\theta = -\dfrac{\sqrt{2}}{2}$
   It can then be deduced that $\theta = -\dfrac{\pi}{4} \quad [2\pi]$, hence :
$$z = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$
2) Find Cartesian form of $z = \sqrt{3}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$
   We find that $z = \sqrt{3}\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{3}}{2} + \dfrac{3}{2}i$

4.1.2 Properties of the modulus and argument

Property 2 : For all complex numbers z other than 0, the following relations apply :
$$|-z| = |z| \quad \text{and} \quad \arg(-z) = \arg(z) + \pi \quad [2\pi]$$
$$|\bar{z}| = |z| \quad \text{et} \quad \arg(\bar{z}) = -\arg(z) \quad [2\pi]$$

Theorem 4 : For all complex numbers z and z' other than 0, the following relations apply :
$$|z\,z'| = |z|\,|z'| \quad \text{and} \quad \arg(z\,z') = \arg(z) + \arg(z') \quad [2\pi]$$
$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n\arg(z) \quad [2\pi]$$
$$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|} \quad \text{and} \quad \arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z') \quad [2\pi]$$

Proof : $z = r(\cos\theta + i\sin\theta)$ and $z' = r'(\cos\theta' + i\sin\theta')$. we deduce that :
$$\begin{aligned} z\,z' &= r\,r'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= r\,r'(\cos\theta\cos\theta' + i\cos\theta\sin\theta' + i\sin\theta\cos\theta' - \sin\theta\sin\theta') \\ &= r\,r'(\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\cos\theta\sin\theta' + \sin\theta\cos\theta') \\ &= r\,r'(\cos(\theta+\theta') + i\sin(\theta+\theta')) \end{aligned}$$
By identification, it can then be deduced that :
$$|z\,z'| = r\,r' = |z|\,|z'| \quad \text{and} \quad \arg(z\,z') = \arg(z) + \arg(z') \quad [2\pi]$$
The equalities $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$ are proven by induction of the properties of the product.

As for the quotient, let $Z = \dfrac{z}{z'}$, therefore $z = Z \times z'$, and according to the properties of the product we find that :

$|z| = |Z| \times |z'| \quad \Leftrightarrow \quad |Z| = \dfrac{|z|}{|z'|}$

$\arg(z) = \arg(Z) + \arg(z') \quad [2\pi] \quad \Leftrightarrow \quad \arg(Z) = \arg(z) - \arg(z') \quad [2\pi]$

4.2 Exponential form

4.2.1 Definition

Let us define the function f whose domain is $\mathbb{R}$ and codomain is $\mathbb{C}$ such that : $f(\theta) = \cos\theta + i\sin\theta$.

Let us calculate $f(\theta)f(\theta')$

$$\begin{aligned}
f(\theta)f(\theta') &= (\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\
&= (\cos\theta\cos\theta' + i\cos\theta\sin\theta' + i\sin\theta\cos\theta' - \sin\theta\sin\theta') \\
&= (\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\cos\theta\sin\theta' + \sin\theta\cos\theta') \\
&= (\cos(\theta+\theta') + i\sin(\theta+\theta')) \\
&= f(\theta+\theta')
\end{aligned}$$

We therefore find that $f(\theta+\theta') = f(\theta)f(\theta')$. This is a characteristic property of exponential function. Indeed, the only functions differentiable in $\mathbb{R}$ which transform a sum into a product are those that satisfy $f(x) = e^{kx}$ or else the zero function. Here we have $f(0) = \cos 0 = 1$, so f cannot be the zero function , we therefore have $f(x) = e^{kx}$

Let's take the derivative of the function f to determine k :

$$\begin{aligned}
f'(\theta) &= -\sin\theta + i\cos\theta \\
&= i^2\sin\theta + i\cos\theta \\
&= i(\cos\theta + i\sin\theta) \\
&= if(\theta)
\end{aligned}$$

we then find $k = i$ because $\left(e^{kx}\right)' = ke^{kx}$
We obtain the following equality from these two properties : $e^{i\theta} = \cos\theta + i\sin\theta$.

Definition 4 : The exponential form of a complex number $z \neq 0$, is :

$$z = re^{i\theta} \qquad \text{with} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)\ [2\pi]$$

Note : We can now sit back and admire the expression : $e^{i\pi} + 1 = 0$.

This expression contains the most important numbers in the history of mathematics :
- 0 and 1 for arithmetic
- π for geometry
- e for analysis
- i for complex numbers

5 Complex numbers and vectors

5.1 Definition

Definition 5 : Consider the complex plane with an orthonormal basis $(O, \overrightarrow{u}, \overrightarrow{v})$, the following relations apply to point $M(z)$

$$z_{\overrightarrow{OM}} = z \quad \text{and} \quad OM = |z| \quad \text{and} \quad (\overrightarrow{u}, \overrightarrow{OM}) = \arg(z)$$

5.2 Affix of a vector

For $A(z_A)$ and $B(z_B)$, we have : $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \quad \Leftrightarrow \quad z_{\overrightarrow{AB}} = z_B - z_A$

Law 2 : For all points A et B in the complex plane :

$$z_{\overrightarrow{AB}} = z_B - z_A \qquad AB = |z_B - z_A| \qquad (\overrightarrow{u}, \overrightarrow{AB}) = \arg(z_B - z_A)$$

Example : Given : $A(2+i)$ and $B(-1-2i)$, determine the coordinates of the vector $\overrightarrow{AB}$, the distance AB and the angle $(\vec{u}, \overrightarrow{AB})$.

- We have : $z_{\overrightarrow{AB}} = z_B - z_A = -1-2i-2-i = -3-3i$ then $\overrightarrow{AB} = (-3;-3)$
- We have : $AB = |z_B - z_A| = \sqrt{9+9} = 3\sqrt{2}$ then $AB = 3\sqrt{2}$
- We have : $\left.\begin{array}{l} \cos\theta = -\dfrac{3}{3\sqrt{2}} = -\dfrac{\sqrt{2}}{2} \\ \sin\theta = -\dfrac{3}{3\sqrt{2}} = -\dfrac{\sqrt{2}}{2} \end{array}\right\} \theta = -\dfrac{3\pi}{4}\ [2\pi]$ then $(\vec{u}, \overrightarrow{AB}) = -\dfrac{3\pi}{4}\ [2\pi]$

5.3 Set of points

Let us determine a set $\mathscr{E}$ of points M satisfying a property with the affix z of M.

- $|z - z_A| = r$ with $r > 0 \quad \Leftrightarrow \quad AM = r$

  $\mathscr{E}$ is the circle centered at A and of radius r

- $|z - z_A| = |z - z_B| \quad \Leftrightarrow \quad AM = BM$

  $\mathscr{E}$ is the perpendicular bisector of segment [AB]

5.4 Sum of two vectors

Theorem 5 : Let $\overrightarrow{u_1}\,(z_1)$, $\overrightarrow{u_2}\,(z_2)$ and $\overrightarrow{u_3}\,(z_3)$ be given such that :

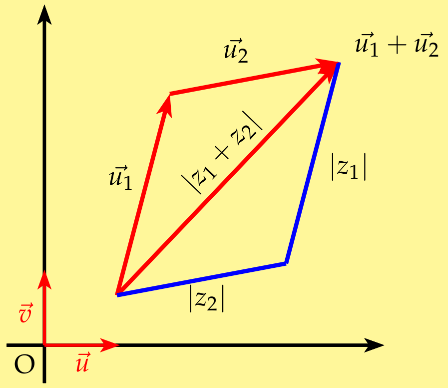
$$\overrightarrow{u_3} = \overrightarrow{u_1} + \overrightarrow{u_2}$$

we therefore deduce that :

$$z_3 = z_1 + z_2$$

and the triangular inequality :

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$



5.5 Oriented angles

Theorem 6 : For all points A, B, C and D such that (A≠B) and (C≠D), we have the following equality :

$$(\overrightarrow{AB}, \overrightarrow{CD}) = \arg\left(\frac{z_D - z_C}{z_B - z_A}\right)$$

Proof : According to the rules governing oriented angles :

$$(\vec{v}, \vec{u}) = -(\vec{u}, \vec{v}) \quad \text{and} \quad (\vec{u}, \vec{w}) = (\vec{u}, \vec{v}) + (\vec{v}, \vec{w})$$

we can therefore deduce the following :

$$\begin{aligned}(\overrightarrow{AB}, \overrightarrow{CD}) &= (\overrightarrow{AB}, \overrightarrow{u}) + (\overrightarrow{u}, \overrightarrow{CD}) \\ &= (\overrightarrow{u}, \overrightarrow{CD}) - (\overrightarrow{u}, \overrightarrow{AB}) \\ &= \arg(z_{\overrightarrow{CD}}) - \arg(z_{\overrightarrow{AB}}) \\ &= \arg(z_D - z_C) - \arg(z_B - z_A) \\ &= \arg\left(\frac{z_D - z_C}{z_B - z_A}\right)\end{aligned}$$

5.6 Collinearity and orthogonality

Property 3 : Alignment of 3 distinct points or parallelism of two lines

$$\text{A, B, C aligned} \quad \Leftrightarrow \quad \overrightarrow{AB} \text{ and } \overrightarrow{AC} \text{ collinear } (\neq \overrightarrow{0}) \quad \Leftrightarrow \quad \frac{z_C - z_A}{z_B - z_A} \in \mathbb{R}$$

$$\text{(AB) and (CD) parallel} \quad \Leftrightarrow \quad \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ collinear } (\neq \overrightarrow{0}) \quad \Leftrightarrow \quad \frac{z_D - z_C}{z_B - z_A} \in \mathbb{R}$$

If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear then : $(\overrightarrow{AB}, \overrightarrow{AC}) = 0$ or $(\overrightarrow{AB}, \overrightarrow{AC}) = \pi$

We deduce that $\arg\left(\dfrac{z_C - z_A}{z_B - z_A}\right) = 0$ or $\arg\left(\dfrac{z_C - z_A}{z_B - z_A}\right) = \pi$

The same technique is used with the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ for two parallel lines

Property 4 : Proving the orthogonality of two lines. If A≠B and C≠D, then

$$(AB) \perp (CD) \quad \Leftrightarrow \quad \overrightarrow{AB} \cdot \overrightarrow{CD} = 0 \quad \Leftrightarrow \quad \frac{z_D - z_C}{z_B - z_A} \text{ purely imaginary}$$

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are orthogonal then : $(\overrightarrow{AB}, \overrightarrow{CD}) = \dfrac{\pi}{2}$ or $(\overrightarrow{AB}, \overrightarrow{CD}) = -\dfrac{\pi}{2}$

Therefore : $\arg\left(\dfrac{z_D - z_C}{z_B - z_A}\right) = \dfrac{\pi}{2}$ or $\arg\left(\dfrac{z_D - z_C}{z_B - z_A}\right) = -\dfrac{\pi}{2}$

5.7 Triangles in the complex plane

To prove that the triangle ABC is :

- **Isosceles with vertex angle A :** $AB = AC \quad \Leftrightarrow \quad |z_B - z_A| = |z_C - z_A|$
- **Equilateral :** $AB = AC = BC$ or $AB = AC$ and $(\overrightarrow{AB}, \overrightarrow{AC}) = \pm\dfrac{\pi}{3}$

  $\Leftrightarrow \quad |z_B - z_A| = |z_C - z_A| = |z_C - z_B|$

  $\Leftrightarrow \quad |z_B - z_A| = |z_C - z_A|$ et $\arg\left(\dfrac{z_C - z_A}{z_B - z_A}\right) = \pm\dfrac{\pi}{3}$
- **Right-angled at A :** $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \quad \Leftrightarrow \quad \dfrac{z_C - z_A}{z_B - z_A}$ purely imaginary
- **Isoceles right-angled at A :** $AB = AC$ and $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \quad \Leftrightarrow \quad \dfrac{z_C - z_A}{z_B - z_A} = \pm i$